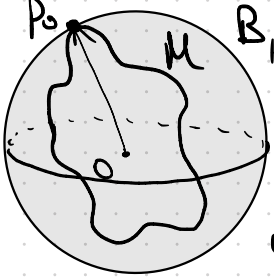
1) WLOG we can translate M so that it contains the origin O.
Consider the function $f: M \to \mathbb{R}$ by $f(p) = |p|^2$ viewing the point p as a vector in $\mathbb{R}^3$ and taking its norm. Then since M is compact, f achieves maximum on M at some point, say p_0.



$B_{|p|}(0)$. Let α be a regular curve parametrized by arc-length s.t. $\alpha(0) = p_0$.
Then since f attains maximum at p_0, by the first and second derivative test, we have

$$0 = \frac{d}{dt} f(\alpha(t))\Big|_{t=0} = 2\langle \alpha'(0), \alpha(0)\rangle \Rightarrow \text{at } p_0,\ \overset{\text{unit normal}}{N(p_0)} = \frac{\alpha(0)}{|\alpha(0)|}$$

$$0 \geq \frac{d^2}{dt^2} f(\alpha(t))\Big|_{t=0} = \langle \alpha''(0), \alpha(0)\rangle + \underbrace{\langle \alpha'(0), \alpha'(0)\rangle}_{=1}$$
$$= \langle \alpha''(0), \alpha(0)\rangle + 1.$$

Now take $\alpha = \gamma_i$ where $\gamma_i(0) = p_0$, $\gamma_i'(0) = v_i$ the ith principal direction ($i = 1, 2$).

Then we get $0 \geq \langle \gamma_i''(0), |\gamma_i(0)| N(p_0)\rangle + 1$
$$= |\gamma_i(0)| \langle -dN_{p_0}(\gamma_i'(0)), \gamma_i'(0)\rangle + 1$$
$$= |\gamma_i(0)| \langle S_{p_0}(v_i), v_i\rangle + 1$$
$$= |\gamma_i(0)| k_i \langle v_i, v_i\rangle + 1$$

$$\Rightarrow 1 + k_i |\gamma_i(0)| \leq 0.$$

$$\Rightarrow k_i \leq \frac{-1}{|\gamma_i(0)|} < 0. \quad \Rightarrow K = k_1 k_2 > 0.$$

2) Define $F: C \to C$ by $F(x,y,z) = (x,-y,-z)$.

- We verify that $F(x,y,z) \in C$:

  Clearly $x^2 + (-y)^2 = x^2 + y^2 = 1$. ✓.

- Clearly F is a diffeomorphism.

- Let us parametrize the cylinder by

$X(u,v) = (\cos u, \sin u, v)$ and by $\bar{X}(s,t) = (\cos s, \sin s, t)$
with the coordinate change under F by

$$(u,v) \to (s,t) = (-u,-v).$$

Then the Jacobian $= \begin{pmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow dF(X_u) = -\bar{X}_u$, $dF(X_v) = -\bar{X}_v$. from which we can see that F is an isometry.

Let (x,y,z) be a fixed point under F, i.e.

$$(x,y,z) = (x,-y,-z) \text{ with } x^2+y^2=1.$$

Then $\begin{matrix} y = -y \\ z = -z \end{matrix} \Rightarrow \begin{matrix} y = 0 \\ z = 0. \end{matrix}$ and we have $x^2 = 1$
$\Rightarrow x = \pm 1$.

So the fixed pts under F are $(1,0,0)$, $(-1,0,0)$

3) a) $g_{ij} = \exp(2f)\delta_{ij} \Rightarrow g^{ij} = \exp(-2f)\delta_{ij}$

Then $\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right)$

$$= \frac{1}{2}\sum_{l=1}^{2} \exp(-2f)\delta_{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right)$$

$$= \frac{1}{2}\exp(-2f)\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right)$$

$$= \frac{1}{2}\exp(-2f)\left(\frac{\partial(e^{2f}\delta_{ik})}{\partial x^j} + \frac{\partial(e^{2f}\delta_{jk})}{\partial x^i} - \frac{\partial(e^{2f}\delta_{ij})}{\partial x^k}\right)$$

Note: $\frac{\partial}{\partial x^j}\left(e^{2f}\delta_{ik}\right) = e^{2f}\cancel{\left(\frac{\partial}{\partial x^j}\delta_{ik}\right)}^{0} + \frac{\partial(e^{2f})}{\partial x^j}\delta_{ik}$

$$= 2e^{2f}\frac{\partial f}{\partial x^j}\delta_{ik} = 2e^{2f}f_j\delta_{ik}.$$

$$= \cancel{\tfrac{1}{2}}\cancel{e^{-2f}}\cdot\cancel{2}\cancel{e^{2f}}\left(f_j\delta_{ik} + f_i\delta_{jk} - f_k\delta_{ij}\right)$$

$= f_j\delta_{ik} + f_i\delta_{jk} - f_k\delta_{ij}$ as required.

b) Note $\delta_{ij}\delta_{ji} = \sum_{i=1}^{2}\sum_{j=1}^{2} \delta_{ij}\delta_{ji} = 2.$

$$\delta_{ij}\delta_{jk} = \sum_{j=1}^{2} \delta_{ij}\delta_{jk} = \delta_{ik}$$

By Gauss Formula. $K = \frac{1}{2} g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{lk}\Gamma^l_{ji} - \Gamma^k_{lj}\Gamma^l_{ki}\right)$

$$\Gamma^k_{ij,k} = \frac{\partial}{\partial x^k}\left(f_j\delta_{ik} + f_i\delta_{jk} - f_k\delta_{ij}\right)$$

$$= f_{jk}\delta_{ik} + f_{ik}\delta_{jk} - f_{kk}\delta_{ij} = f_{ji} + f_{ij} - f_{kk}\delta_{ij}$$

So $g^{ij}\left(\Gamma^k_{ij,k}\right) = e^{-2f}\delta_{ij}\left(2f_{ij} - f_{kk}\delta_{ij}\right)$

$= e^{-2f}\left(2f_{ij}\delta_{ij} - f_{kk}\delta_{ij}\delta_{ij}\right)$

$= e^{-2f}\left(2f_{ii} - 2f_{kk}\right) = 0$

Summing over k.

$$\Gamma^k_{ik,j} = \frac{\partial}{\partial x^j}\left(\cancel{f_k\delta_{ik}} + f_i\delta_{kk} - \cancel{f_k\delta_{ik}}\right) = f_{ij}\delta_{kk} \overset{\downarrow}{=} 2f_{ij}$$

So $g^{ij}\left(\Gamma^k_{ik,j}\right) = 2e^{-2f}f_{ij}\delta_{ij} = 2e^{-2f}\Delta f.$

$$\Gamma^k_{lk}\Gamma^l_{ji} = \left(\cancel{f_k\delta_{lk}} + f_l\delta_{kk} - \cancel{f_k\delta_{lk}}\right)\left(f_i\delta_{jl} + f_j\delta_{il} - f_l\delta_{ji}\right)$$

$$= 4f_jf_i - 2f_lf_l\delta_{ji}$$

So $g^{ij}\left(\Gamma^k_{lk}\Gamma^l_{ji}\right) = e^{-2f}\left(4f_jf_i\delta_{ij} - 2f_lf_l\delta_{ji}\delta_{ij}\right) = 0.$

$$\Gamma^k_{lj}\Gamma^l_{ki} = (f_j\delta_{lk} + f_l\delta_{jk} - f_k\delta_{lj})(f_i\delta_{kl} + f_k\delta_{il} - f_l\delta_{ki})$$

$$= 2f_jf_i + f_jf_k\delta_{ki} - f_jf_l\delta_{li} + f_lf_i\delta_{jl} + f_lf_k\delta_{jk}\delta_{il}$$

$$- f_lf_l\delta_{ji} - f_kf_i\delta_{jk} - f_kf_k\delta_{ji} + f_kf_l\delta_{lj}\delta_{ki}$$

$$= 2f_jf_i - 2f_lf_l\delta_{ji} + 2f_lf_k\delta_{jk}\delta_{il}$$

So $g^{ij}(\Gamma^k_{lj}\Gamma^l_{ki}) = e^{-2f}(2f_if_i - 4f_lf_l + 2f_lf_k\delta_{kl}) = 0.$

So finally we have

$$K = \frac{1}{2}g^{ij}(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{lk}\Gamma^l_{ji} - \Gamma^k_{lj}\Gamma^l_{ki})$$

$$= -\frac{1}{2}g^{ij}\Gamma^k_{ik,j} = -e^{-2f}\Delta f.$$

4) a) $g_{ij} = \frac{4}{(1+|u|^2)^2}\delta_{ij}$ $\quad u=(u_1,u_2)$.

Note if we let f be s.t. $e^{2f} = \frac{4}{(1+|u|^2)^2}$ $(*)$, then g has the structure as in the previous question above, and we can more easily compute K.

$$(*) \Rightarrow f = \log\left(\frac{2}{1+|u|^2}\right) = \log\left(\frac{2}{1+(u_1^2+u_2^2)}\right)$$

$$\frac{\partial f}{\partial u_1} = \frac{-2u_1}{1+|u|^2}$$

$$\frac{\partial^2 f}{\partial u_1^2} = \frac{2(u_1^2-u_2^2-1)}{(1+|u|^2)^2}.$$

$$\frac{\partial f}{\partial u_2} = \frac{-2u_2}{1+|u|^2}, \quad \frac{\partial^2 f}{\partial u_2^2} = \frac{2(u_2^2-u_1^2-1)}{(1+|u|^2)^2}$$

$$\Rightarrow \Delta f = \frac{-4}{(1+|u|^2)^2}$$

$$e^{-2f} = e^{-2\log\left(\frac{2}{1+|u|^2}\right)} = e^{\log\left(\frac{(1+|u|^2)^2}{4}\right)} = \frac{(1+|u|^2)^2}{4}.$$

So $K = -e^{-2f}\Delta f = \cancel{-}\left(\cancel{\frac{(1+|u|^2)^2}{4}}\right)\left(\frac{\cancel{-4}}{\cancel{(1+|u|^2)^2}}\right) = 1$

b) Similarly, letting $e^{2f} = \dfrac{4}{(1-|u|^2)^2}$, $|u|^2 \neq 1$,

we got $f = \begin{cases} \log\left(\dfrac{2}{1-|u|^2}\right) & |u|^2 < 1 \\ \log\left(\dfrac{2}{|u|^2-1}\right) & |u|^2 > 1 \end{cases}$, $e^{-2f} = \dfrac{(1-|u|^2)^2}{4}$

$\dfrac{\partial f}{\partial u_i} = \dfrac{2u_i}{1-|u|^2}$, $\dfrac{\partial^2 f}{\partial u_1^2} = \dfrac{-2u_1^2+2u_2^2+2}{(1-|u|^2)^2}$, $\dfrac{\partial^2 f}{\partial u_2^2} = \dfrac{-2u_2^2+2u_1^2+2}{(1-|u|^2)^2}$

$i = 1, 2.$

$\Rightarrow \Delta f = \dfrac{4}{(1-|u|^2)^2}$

So $K = -e^{-2f}\Delta f = -\dfrac{\cancel{(1-|u|^2)^2}}{\cancel{4}}\dfrac{\cancel{4}}{\cancel{(1-|u|^2)^2}} = -1$

c) let $g_{ij} = \dfrac{4}{(k+|u|^2)^2}\delta_{ij}$, $k \neq 0$, then similar to above,

let $f = \log\left|\dfrac{2}{k+|u|^2}\right|$ yields $e^{-2f} = \dfrac{(k+|u|^2)^2}{4}$, $\dfrac{\partial f}{\partial u_i} = \dfrac{-2u_i}{k+|u|^2}$

$i = 1, 2$

$\dfrac{\partial^2 f}{\partial u_1^2} = \dfrac{-2(k-u_1^2+u_2^2)}{(k+|u|^2)^2}$, $\dfrac{\partial^2 f}{\partial u_2^2} = \dfrac{-2(k-u_2^2+u_1^2)}{(k+|u|^2)^2}$, $\Rightarrow \Delta f = \dfrac{-4k}{(k+|u|^2)^2}$

So $K = -e^{-2f}\Delta f = \cancel{-}\dfrac{\cancel{(k+|u|^2)^2}}{\cancel{4}}\left(\dfrac{\cancel{-4}k}{\cancel{(k+|u|^2)^2}}\right) = k.$

5) $X(u,v) = (u\cos v, u\sin v, \log u)$

$\bar{X}(u,v) = (u\cos v, u\sin v, v)$

$X_u = (\cos v, \sin v, \frac{1}{u})$

$X_v = (-u\sin v, u\cos v, 0)$

$E = \langle X_u, X_u \rangle = \frac{1}{u^2} + 1$

$F = 0$

$G = u^2$.

$|X_u \times X_v|^2 = EG - F^2 = 1 + u^2$. $\Rightarrow |X_u \times X_v| = \sqrt{1+u^2}$.

$X_u \times X_v = (-\cos v, -\sin v, u)$.

$X_{uu} = (0, 0, \frac{-1}{u^2})$

$X_{uv} = (-\sin v, \cos v, 0)$

$X_{vv} = (-u\cos v, -u\sin v, 0)$.

$e = \langle N, X_{uu} \rangle = \frac{u}{\sqrt{1+u^2}} \cdot \left(\frac{-1}{u^2}\right) = \frac{-1}{u\sqrt{1+u^2}}$.

$f = \langle N, X_{uv} \rangle = 0$

$g = \langle N, X_{vv} \rangle = \frac{1}{\sqrt{1+u^2}} \left(u\cos^2 v + u\sin^2 v\right) = \frac{u}{\sqrt{1+u^2}}$

So $K = \frac{eg - f^2}{EG - F^2} = \frac{eg}{EG} = \frac{\frac{-1}{1+u^2}}{1+u^2} = \frac{-1}{(1+u^2)^2}$

$$\bar{X}(u,v) = (u\cos v,\ u\sin v,\ v)$$

$$\bar{X}_u = (\cos v, \sin v, 0)$$

$$\bar{X}_v = (-u\sin v, u\cos v, 1)$$

$$\bar{E} = \langle \bar{X}_u, \bar{X}_u \rangle = 1,\quad \bar{F} = 0,\quad \bar{G} = 1+u^2$$

$$|\bar{X}_u \times \bar{X}_v|^2 = \bar{E}\bar{G} - \bar{F}^2 = 1+u^2 \Rightarrow |\bar{X}_u \times \bar{X}_v| = \sqrt{1+u^2}$$

$$\bar{X}_u \times \bar{X}_v = (\sin v, -\cos v, u)$$

$$\bar{X}_{uu} = (0,0,0) \qquad \bar{e} = \langle \bar{N}, \bar{X}_{uu} \rangle = 0$$

$$\bar{X}_{uv} = (-\sin v, \cos v, 0) \qquad \bar{f} = \langle \bar{N}, \bar{X}_{uv} \rangle = \frac{-1}{\sqrt{1+u^2}}$$

$$\bar{X}_{vv} = (-u\cos v, -u\sin v, 0) \qquad \bar{g} = \langle \bar{N}, \bar{X}_{vv} \rangle = 0.$$

$$\bar{K} = \frac{\bar{e}\bar{g} - \bar{f}^2}{\bar{E}\bar{G} - \bar{F}^2} = \frac{\frac{-1}{1+u^2}}{1+u^2} = \frac{-1}{(1+u^2)^2} = K.$$

So $K = \bar{K}$ but they do not have the same first fundamental form.